JOURNAL OF ALGEBRA 56, 91–102 (1979)

On the Moduli of Plane Curve Singularities, I

SHERWOOD WASHBURN

Brandeis University, Waltham, Massachusetts 02154

Communicated by D. Buchsbaum

Received November 25, 1975

The purpose of this paper is to begin a study of certain families of rings of the form $B = k[[X, Y]]/(f)$, where k is an algebraically closed field of characteristic zero and $f(X, Y)$ is an irreducible power series. If $k = \mathbb{C}$, a classical theory describes these rings in terms of the "local knot" of the corresponding plane curve singularity ([1], [4], [6], [10]): for example by showing that the local knot determines, and is determined by, the set of characteristic pairs of B. This knot type is now called the "equisingular type" of B: it is simply the appropriate notion of topological type, the notion analogous to the genus of a non-singular plane curve. More recent work has given conditions for the topological type to be preserved under specialization, that is, for a flat family $B_T = k[[X, Y, T]]/(f(X, Y, T))$ to be "equisingular" ([11], [12]; see also [8]).

Recent work of O. Zariski ([11], [12]) has attracted attention to the problem of giving a satisfactory description of the space of plane branches of a given equisingular type. This problem, the local analog of the moduli problem for non-singular curves of a given genus, is both interesting in itself and related via specialization to the global moduli problem. It is natural to introduce the techniques of deformation theory: studies of the problem from this point of view include [2], [3], [5], [9], and [12A]. The deepest results have been obtained by M. Merle in [3], who shows that the space $\mathcal{M}_\Gamma$ of isomorphism classes of branches (including branches of higher codimension) with a given semigroup Γ admits a natural finite stratification into algebraic varieties $\mathcal{M}_\tau$, which is also the coarsest stratification into separated subspaces (this was conjectured by D. Mumford). The deformation theory here is based on M. Lejeune's construction (see [12A]) of a "special ring" for branches B with a given semi-group Γ: one has $gr_m B = C[t^{\beta_0}, \ldots, t^{\beta_g}]$, where the β_i are generators of the semi-group Γ, for *any* such B, so any such B is a Γ-constant deformation of $\mathbb{C}[t^{\beta_i}]$ (using D. Rim's observation that B is a flat deformation of any associated graded).

Two related difficulties have so far forced descriptions of the space $\mathcal{M}_\Gamma$ to be very indirect. First, the "tangent space" to the equisingular deformation functor is hard to describe (for a plane branch); and second, the defining equation of a plane branch is hard to write down efficiently (this has only been done

0021-8693/79/010091-12$02.00/0

in the case of a single characteristic pair [9, Prop. 6.6], [12, p. 99]). It follows from Lemmas 1.5ii, 1.7, and Theorem 2.3 below that if $k[[X, Y]]/(f)$ is an irreducible branch with characteristic pair set $[(m_i, n_i)]_{i=1}^g$ there is a finite sequence of polynomials $f_1(X, Y),..., f_g(X, Y)$ and an isomorphism $k[[X, Y]]/(f) = k[[X, Y]]/(f_g)$, where $f_1(X, Y) = Y^{n_1} - X^{m_1} + \sum v_{ij}^{(1)} X^i Y^j$ and $f_k(X, Y) = f_{k-1}(X, Y)^{n_k} + \sum v_{ij}^{(k)} X^i Y^j$ for $k = 2,..., g$. This is a form of the Weierstrauss Preparation Theorem for functions of two variables; at present no assertion is made about which terms $v_{ij}^{(k)}$ can occur. What is needed is a "Versal Preparation Theorem" which identifies a minimal set of $v_{ij}^{(k)}$ occurring in the above expression. Such a theorem will be proved in a later paper in this series. There seems to be a whole class of strong preparation theorems of this nature: in fact one can ask whether for an arbitrary power series $f(X_1,..., X_n)$ it is possible to define a "characteristic" and to prove a preparation theorem which puts $f(X_1,..., X_n)$ in Weierstrauss form in a manner which depends on the characteristic.

Section 1 below consists of an exposition of some needed properties of plane branches over rings. The necessary information can be summarized in the specialization properties of these branches over suitable base rings: in particular a flatness criterion for branches over an artin ring can be obtained in this way. The results are summarized in Lemma 1.5 below. Irreducible plane branches have an "underlying multiple branch structure" (Example 1.4) which can only really be understood by working over a base ring.

Section 2 below contains the existence Theorem 2.3 which asserts that a plane branch over a power series ring has an essentially unique versal equisingular deformation whose base ring is again a power series ring (compare Theorems 3.2 and 4.2 of [9]). The proof is a straightforward application of the Theorem of Schlessinger [7]. The novel feature is that the manner in which equisingular deformations are introduced (they are the "E-deformations" of Section 2) gives rise to a new formula for the number of parameters occurring in the versal equisingular deformation.

1. Branches over Rings

Throughout this paper, k will denote an algebraically closed field of characteristic zero. All rings A, B, etc. will be commutative with unit and all modules unitary; unless otherwise stated A, B, etc. will be complete local noetherian k-algebras with residue field k. Modules over A, B etc. will be complete in the adic topology and homomorphisms of these modules will be understood to be continuous.

Let t be an indeterminant. An element $t_A = a_1 t + a_2 t^2 + \cdots$ of $A[[t]]$ is a *parameter* if $a_1 \in A^*$ or equivalently if t_A is the image of t under an A-algebra automorphism of $A[[t]]$.

In this paper rings of the form $B = A[[t_A{}^n, y(t_A)]]$ will be considered, where $t_A \in A[[t]]$ is a parameter. If $y(t_A) = a_m t_A{}^m + a_{m+1} t_A^{m+1} + \cdots$, let $g(B) =$ g.c.d.$(n, [i\colon a_i \neq 0])$.

DEFINITION 1.1. A *parametrized branch over A with parameter t_A* is an A-algebra of the form

$$B_A = A[[t_A{}^n, y(t_A)]] \subset A[[t]]$$

where $t_A \in A[[t]]$ is a parameter and $y(t_A) = t_A^{\beta_1} + \sum_{i>\beta_1} a_i t_A{}^i$, $\beta_1 > n$, n does not divide β_1, and $g(B_A) = 1$.

A subscript will be used to indicate the base ring, as in the definition, throughout this paper.

Remarks 1.2. If $A = k$, it is well known that every domain of the form $k[[X, Y]]/(f)$ is a parametrized branch for a suitable parameter. If $B_A = A[[X, Y]]/\mathfrak{A}$ and A is factorial, $\mathfrak{A}$ is principal, but this is not the case in general.

CHARACTERISTICS 1.3. A parametrized branch B has a *characteristic* $I(B)$ defined in the usual way as follows. Let $e_0 = n$, and $e_1 =$ g.c.d.$(e_0, \beta_1) < n$. Let $n = n_1 e_1$, and $\beta_1 = m_1 e_1$. If n_i, m_i, e_i, and β_i have been defined, let β_{i+1} be the smallest integer with $a_{\beta_{i+1}} \neq 0$ and $e_{i+1} =$ g.c.d.$(e_i, \beta_{i+1}) < e_i$ ($a_{\beta_{i+1}} t^{\beta_{i+1}}$ is then a *characteristic term*). Let $\beta_{i+1} = m_{i+1} e_{i+1}$ and $e_i = n_{i+1} e_{i+1}$. Since the sequence $e_0 > e_1 > \cdots$ is strictly decreasing one has $e_g = 1$ for some minimal g. The set of integers $I(B) = [n; \beta_1, ..., \beta_g]$ is the characteristic of the branch B. The set $[(m_i, n_i)]_{i=1}^g$ is the set of *characteristic pairs* of B; this set of pairs of integers and the characteristic determine each other. A straightforward argument shows that the characteristic is not altered by change of parameter.

If $A = k$, the *equisingular type* of B is given by the characteristic $I(B)$; by the characteristic pair set; or any data equivalent to these.

An important construction of Zariski gives a "universal" ring of a given characteristic. If $I = [n; \beta_1, ..., \beta_g]$ is a characteristic let

$$B[I, t) = k[[t^n, t^{\beta_1 + i_1 e_1}, t^{\beta_2 + i_2 e_2}, ..., t^{\beta_g + i_g e_g}]]$$

where for $1 \leqslant j \leqslant g$, $i_j \geqslant 0$. $B(I, t)$ is the *Zariski saturation* of any plane branch $B = k[[t^n, y(t)]]$ with characteristic I with respect to the *transversal parameter* t^n (parameter of minimal value n) [11 III]. Theorem 1.8 and Theorem 1.12 of [11 III] assert that the saturation is determined as above as a function of the characteristic and parameter. $B(I, t)$ is the unique minimal subalgebra of $k[[t]]$ such that there is a projection $\mathrm{Spec}(B(I, t)) \twoheadrightarrow \mathrm{Spec}(B)$ for any irreducible branch B with the given parameter and characteristic. For a suitably parametrized family, equisingularity is equivalent to *equisaturation* (triviality of the saturation of the family along a suitable section: see Section 2 below); and this fact can be exploited to define equisingular deformations.

Example 1.4. Consider the parametrized branches

$$
\begin{aligned}
B_0 &= k[[t^2, t^3]] = k[[X, Y]]/(f_0)\\
B_1 &= k[[S_2, t^4, t^6 + S_2 t^7]] = k[[S_2, X, Y]]/(f_1)\\
B_2 &= k[[S_2, S_3, t^8, t^{12} + S_2 t^{14} + S_3 t^{15}]] = k[[S_2, S_3, X, Y]]/(f_2)
\end{aligned}
$$

These are the simplest parametrized branches with one, two, and three characteristic pairs, made into families by varying the characteristic terms. One has

$$
\begin{aligned}
f_0 &= Y^2 - X^3\\
f_1 &= (Y^2 - X^3)^2 - 4S_2^2 X^5 Y - S_2^4 X^7\\
f_2 &= ((Y^2 - X^3)^2 - 4S_2^2 X^5 Y - (12 S_2 S_3^2 + S_2^4) X^7)^2\\
&\quad - 32 S_2 S_3^2 X^{10}(Y^2 - X^3) - (96 S_2^3 S_3^2 + 16 S_3^4) X^{12} Y\\
&\quad - (16 S_2^3 S_3^2 + 8 S_3^4)\, X^e Y (Y^2 - X^3) - (152 S_2^2 S_3^4 + 16 S_2^5 S_3^2) X^{14}\\
&\quad - 20 S_2^2 S_3^4 X^{11}(Y^2 - X^3) - 8 S_2 S_3^6 X^{13} Y - S_3^8 X^{15}.
\end{aligned}
$$

Clearly $B_2/(S_3)B_2 = k[[S_2, X, Y]]/(f_1^2)$; and $B_2/(S_2, S_3)B_2 = k[[X, Y]]/(f_0^4)$. The example suggests that plane branches of a given characteristic can be systematically viewed as deformations of multiple branches of a simpler type.

If $B_A = A[[t_A^n, y(t_A)]]$ is a parametrized branch over a power series ring $A = k[[T_1, ..., T_r]]$ then the integral closure of B_A in its quotient field is $A[[t_A]]$ because t_A is integral over B_A and $A[[t_A]]$ is formally smooth. Since $\dim(B_A) = \dim(\bar{B}_A) = r + 1$ and $\dim(A[[X, Y]]) = r + 2$, and $A[[X, Y]]$ is factorial, $B_A = A[[X, Y]]/(f)$. Since $y(t_A)$ is integral over $A[[t_A^n]]$ one can assume that $f = Y^m + a_{m-1}Y^{m-1} + \cdots + a_1 Y + a_0 \in A[[X]][Y]$ is an integral polynomial in Y. It follows easily from the irreducibility assertion in Lemma 1.5 that $m = n$ and that Y^n is the leading form of f. For $1 \leqslant s \leqslant r$, let $K_{0,s}$ be the quotient field of $k[[T_1, ..., T_s]]$, and K_s the quotient field of $k[[T_1, ..., T_s, X]]$. Let $i_s : k[[T_1, ..., T_r, X]][Y] \to K_s \hat{\otimes}_{k[[T_1, ..., T_s, X]]} k[[T_1, ..., T_r, X]][Y]$ be the natural map, where the completion is the $(T_{s+1}, ..., T_r)$-adic completion.

Lemma 1.5. (i) *$i_s(f)$ is irreducible.*

(ii) *Let $B_A = A[[t_A^n, y(t_A)]]$ be a parametrized branch over $A = k[[T_1, ..., T_r]]$ and $I = (T_{s+1}, ..., T_r)A$; denote reduction* (mod I) *by the subscript "0". Thus $A_0 = k[[T_1, ..., T_s]]$ and $B_{A_0} = A_0[[t_{A_0}^n, y_0(t_{A_0})]] = A_0[[X, Y]]/(h)$ for some irreducible $h \in A_0[[X]][Y]$. Then*

$$B_0 = B_A/IB_A = A_0[[X, Y]]/(h^e), \qquad \text{where} \quad e = g(B_{A_0}).$$

(iii) *Let B_A be a parametrized branch over $A = A_1 \otimes_k A_0$, where A_1 is an artin k-algebra and A_0 is a power series ring. Then $B_A/m_{A_1}B_A = A_0[[X, Y]]/(k^{e'})$, where $1 \leqslant e' \leqslant e = g(B_{A_0})$ (here reduction* (mod m_{A_1}) *replaces reduction* (mod I) *in* (ii). *If $B_A = A[[X, Y]]/\mathfrak{A}$, the following are equivalent*:

(1) *$\mathfrak{A}$ is principal.*

(2) $e' = 1$.

(2) *B_A is A-flat.*

Proof of i. The irreducibility of $i_s(f)$ is equivalent to the assertion that $K_s \hat{\otimes}_{k[[T_1,\ldots,T_s,X]]} B_A$ is a domain. Since every element of $k[[T_1, \ldots, T_s, X]]$ becomes invertible in $K_{0,s}((X)) = K_{0,s}((t_A{}^n))$ there is an inclusion $K_s \subset K_{0,s}((t_A{}^n))$. Now

$$\begin{aligned} K_s \hat{\otimes} B_A &= K_s[[T_{s+1}, \ldots, T_r, y(t_A)]] \\ &= K_s[[T_{s+1}, \ldots, T_r]][y(t_A)] \\ &\subset K_{0,s}((t_A{}^n))[[T_{s+1}, \ldots, T_r]][y(t_A)] \\ &\subset K_{0,s}((t_A{}^n))[[T_{s+1}, \ldots, T_r]][t_A] \\ &= K_{0,s}((t_A{}^n))[[T_{s+1}, \ldots, T_r]][Z]/(Z^n - t_A{}^n). \end{aligned}$$

Of course the roots of $Z^n - t_A{}^n$ are $\zeta^i t_A$, where ζ is a primitive n-th root of unity. Since none of these roots are contained in $K_{0,s}((t_A{}^n))[[T_{s+1}, \ldots, T_r]][Z]$, $Z^n - t_A{}^n$ is irreducible in this ring, so $K_s \hat{\otimes} B_A$ is contained in a domain and $i_s(f)$ is irreducible.

Proof of (ii). There is a commutative diagram

$$\begin{array}{ccc} K_s[[T_{s+1}, \ldots, T_r]][Y] & \xrightarrow{/I} & K_s[Y] \\ \Big\downarrow{\scriptstyle i_s} & & \Big\downarrow{\scriptstyle i_0} \\ A[[X]][Y] & \xrightarrow{/I} & A_0[[X]][Y] \end{array}$$

where i_s is as in Lemma 2.1 and i_0 is the obvious inclusion. By (i), $i_s(f)$ is irreducible. Now consider the reduction of f (mod I), $f_0 \in A[[X]][Y]$: clearly $f_0 = h^a g$ where $a \geqslant 1$, h is a monic polynomial in Y, and h does not divide g. Assume that $\deg_Y(g) > 0$: by Gauss' Lemma $i_0(h)$ does not divide $i_0(g)$, and as a result $i_0(h^a)$ and $i_0(g)$ generate the unit ideal in $K_s[Y]$. Then by Hensel's Lemma the factorization $i_0(f_0) = i_0(h^a)\, i_0(g)$ lifts to a factorization of $i_s(f)$, contradicting the irreducibility of $i_s(f)$. Thus g is a unit, and since f_0 and h are monic in Y, $f_0 = h^a$. Since $\deg_Y(h) = n/e$, $f_0 = h^e$.

Proof of (ii). Choose a surjection p: $k[[T_{s+1}, \ldots, T_r]] \to A_1$: this gives a surjection $1 \hat{\otimes} p$: $\mathbf{A} = k[[T_1, \ldots, T_r]] \to A$. Let $B_{\mathbf{A}} = \mathbf{A}[[t_{\mathbf{A}}{}^n, y(t_{\mathbf{A}})]] = \mathbf{A}[[X, Y]]/(f)$ be a parametrized branch over $\mathbf{A}$ with a surjection $B_{\mathbf{A}} \to B_A$

reducing to $1 \mathbin{\widehat{\otimes}} p$ on $\mathbf{A}$. Then $f = h^e + h_1$ with $h_1 \in (T_i)\,\mathbf{A}[[X, Y]]$ by (ii). This gives a surjection $B_{\mathbf{A}}/(T_{s+1},..., T_r)B_{\mathbf{A}} = A_0[[X, Y]]/(h^e) \to B_A/m_{A_1}B_A = A_0[[X, Y]]/\mathfrak{A} \cdot A_0[[X, Y]]$. Since there is also a surjection $B_A/m_{A_1}B_A \to B_{A_1} = A_0[[X, Y]]/(h)$ one has $(h^e)\, A_0[[X, Y]] \subset \mathfrak{A} \cdot A_0[[X, Y]] \subset (h)\, A_0[[X, Y]]$, so $\mathfrak{A} \cdot A_0[[X, Y]] = (h^{e'})\, A_0[[X, Y]]$, $1 \leqslant e' \leqslant e$.

The argument implies that $\mathfrak{A}$ has a basis $\mathfrak{A} = (g, g_1,..., g_t)\, A[[X, Y]]$, where $g = h^{e'} + h_1$, and $h_1, g_1,..., g_t \in m_A \cdot A[[X, Y]]$; and further that $\mathfrak{A}$ is principal if and only if $\mathfrak{A} = (h^{e'} + h_1)\, A[[X, Y]]$.

(1) $\Rightarrow$ (3). If $\mathfrak{A} = (h^{e'} + h_1)\, A[[X, Y]]$ is principal $h^{e'} + h_1 \equiv h^{e'} \not\equiv 0 \pmod{m_{A_1}}$ so B_A is flat by one of the variants of the local criterion.

(1) $\Rightarrow$ (2). Suppose that $e' > 1$. It is clear by inspection of the Taylor series that $h(t_A{}^n, y(t_A)) \in m_{A_1} \cdot A[[t_A]]$ and, since $e' > 1$, the surjection $B_A \to B_{A_0}$ does not split, so $h(t_A{}^n, y(t_A)) \neq 0$. Thus $\eta \cdot h(X, Y) \in \mathfrak{A}$ for some $\eta \in m_{A_1}$. $y(t_A)$ is integral over $A[[t_A{}^n]]$ and therefore satisfies an equation of the form $h^{e'} + h_1 = 0$. Notice that if $\mathfrak{A}$ is principal it must be generated by an integral equation. To see this, let $\mathfrak{A} = (h^{e'} + h_1')\, A[[X, Y]]$. Then $h^{e'} + h_1 = (g_0 + g_1)(h^{e'} + h_1')$ where $g_0 \in A_0[[X, Y]]$ and g_1, $h_1' \in m_{A_1}A[[X, Y]]$. Clearly $g_0 = 1$, so $g_0 + g_1$ is a unit, so $h^{e'} + h_1$ generates $\mathfrak{A}$ with $\deg_Y(h_1) < \deg_Y(h^{e'})$. But then by inspecting the degree in Y of both sides, one sees that an equation of the form $\eta \cdot h(X, Y) = h_2(X, Y)(h(X, Y)^{e'} + h_1(X, Y))$ is impossible. Thus $\mathfrak{A}$ is not principal.

(2) $\Rightarrow$ (1). Suppose that $e' = 1$. Then $\mathfrak{A} = (h + h_1, g_1,..., g_t)\, A[[X, Y]]$, and by induction on $\dim_k(m_{A_1})$ one can assume that $g_i(X, Y) = \eta \cdot g_i'(X, Y)$ where $\eta \cdot m_{A_1} = (0)$, and $g_i'(X, Y) \in A_0[[X, Y]]$. Then $g_i'(t_A{}^n, y(t_A)) \in m_{A_1} \cdot A[[t_A]]$, so $g_i'(X, Y) = h_i'(X, Y)\, h(X, Y)$ for $i = 1,..., t$. Then $\eta \cdot g_i'(X, Y) = \eta \cdot h_i'(X, Y)\, h(X, Y) = \eta \cdot h_i'(X, Y)(h(X, Y) + h_1(X, Y))$ so $\mathfrak{A}$ is principal.

(3) $\Rightarrow$ (1). If $\mathfrak{A}$ is not principal, $\mathfrak{A} = (h^{e'} + h_1, \eta \cdot h,...)$ with $e' > 1$, and $\eta \in \operatorname{ann}(h(t_A{}^n, y(t_A))) = \ell$. Assume further that $\eta \in \ell - \ell^2$. Then $\eta \otimes h(t_A{}^n, y(t_A)) \neq 0 \in m_{A_1} \otimes_{A_1} B_A$: otherwise $h(t_A{}^n, y(t_A)) = \eta_1 b_1 + \cdots + \eta_s b_s$ with $\eta_i \in A_1$ and $b_i \in B_A$, and this is impossible since $e' > 1$. Thus $m_{A_1} \otimes_{A_1} B_A \to m_{A_1}B_A$ is not injective, so B_A is not A_1-flat by the local criterion, hence not A-flat.

Remark 1.6. The flatness criterion is the same if A_0 is replaced by its quotient field: it is in this form that the criterion is used in Theorem 2.3 below.

The simplest example of a parametrized branch over the dual numbers which fails to be flat is $B_{k[\epsilon]} = k[[\epsilon, t^4, t^6 + \epsilon \cdot t^7]] = k[[\epsilon, X, Y]]/((Y^2 - X^3)^2, \epsilon \cdot (Y^2 - X^3))$.

Let $I = [n; \beta_1,..., \beta_g]$ be a characteristic. For the deformation theory in the next section one needs a "special ring" with characteristic I. In order to bring out the underlying multiple branch structure, the coefficients of the characteristic terms should be variables so that one can invoke Lemma 1.5(ii). Let

$S_2, \dots, S_g$ be indeterminates and $A = k[[S_2, \dots, S_g]]$, $B_I(t) = A[[t^n, t^{\beta_1} + S_2 t^{\beta_2} + \cdots + S_g t^{\beta_g}]]$.

LEMMA 1.7. *Let B_A be any parametrized branch over A with characteristic I and characteristic terms $S_i t^{\beta_i}$, $2 \leqslant i \leqslant g$. Then B_A is a flat deformation of $B_I(t)$.*

Proof. If $B_A = A[[t^n, y(t)]]$ and $y(t) = t^{\beta_1} + \sum_{i>\beta_1} a_i t^i$, let $B_T = A[[T, t^n, t^{\beta_1} + a_{\beta_1+1} T t^{\beta_1+1} + a_{\beta_1+2} T^2 t^{\beta_1+2} + \cdots]]$. Clearly there is an isomorphism $A[[T, T^{-1}]] \mathbin{\hat{\otimes}}_A B_A = A[[T, T^{-1}]] \mathbin{\hat{\otimes}}_A B_T$. Apply to $A[[T, T^{-1}]] \mathbin{\hat{\otimes}}_A B_T$ the isomorphism induced by $S_i \mapsto S_i T^{\beta_1 - i}$. Then, $B_T/(T)B_T = B_I(t)$.

2. EQUISINGULAR DEFORMATIONS AND E-DEFORMATIONS

The following notation will be used systematically: if B is an A-algebra this will be indicated by a subscript B_A; and base change will be indicated in the same way $B_{A'} = A' \mathbin{\hat{\otimes}}_A B_A$.

Now let $A = A_1 \mathbin{\hat{\otimes}}_k A_0$ and $B_{A_0} = A_0[[t_{A_0}^n, y(t_{A_0})]]$ a parametrized branch over A_0, where A_1 is a complete local k-algebra with residue field k.

DEFINITION 2.1. A *deformation* of B_{A_0} is a commutative square

$$\begin{array}{ccccc} B_A & \longrightarrow & B_{A_0} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ A & \xrightarrow{/m_{A_1}} & A_0 & \longrightarrow & 0 \end{array} \qquad B_{A_0} = A_0 \otimes_A B_A$$

where $B_A = A[[t_A{}^n, y(t_A)]]$ is a parametrized branch over A.

The deformation is *infinitesimal* if A_1 is an artin k-algebra. Let $I = [n; \beta_1, \dots, \beta_g]$ be the characteristic of B_{A_0}. The deformation is an *E-deformation* if there is an A-algebra monomorphism

$$B_A \to A \mathbin{\hat{\otimes}}_k B(I, t_A) = B(I, t_A)_A$$

(where $B(I, t)$ is the saturation of any branch over k with characteristic I with respect to the transversal parameter t^n).

Remarks 2.2. If A_0 is a power series ring it follows from Lemma 1.5(iii) that B_A is A-flat. If $A_0 = k[[T_1, \dots, T_s]]$ and $A = k[[T_1, \dots, T_r]]$, $s < r$, and B_A is an E-deformation of B_{A_0} it is easy to show that $\operatorname{Spec}(B_A)$ is equisaturated (that $\widetilde{\bar{K} \mathbin{\hat{\otimes}}_A B_A} = \widetilde{\bar{K} \mathbin{\hat{\otimes}}_{A_0} B_{A_0}}$ where the saturations are with respect to $t_A{}^n$ and $t_{A_0}^n$ and $\bar{K}$ is the algebraic closure of the quotient field of A): hence (Cor. 7.5 of [11 III]) B_A is equisingular. Thus, E-deformations with A_1 artinian are "infinitesimal equisingular deformations."

Lemma 1.6 asserts that *every* parametrized branch with sufficiently general characteristic terms is an E-deformation of $B_l(t) = k[[S_2,\ldots, S_g, t^n, t^{\beta_1} + S_2 t^{\beta_2} + \cdots + S_g t^{\beta_g}]]$.

If B_{A_0} is a parametrized branch over A_0 and $A = A_1 \mathbin{\widehat{\otimes}_k} A_0$, let

$$E(B_{A_0}, A) = \text{the set of isomorphism classes of } E\text{-deformations of } B_{A_0} \text{ over } A.$$

Here it is understood that the isomorphisms reduce to the identity (mod m_{A_1}). If B_{A_0} is fixed the abbreviation $E(B_{A_0}, A) = E(A)$ will be used. $E(\_)$ is to be regarded as a functor as follows. If $A' = A_1' \mathbin{\widehat{\otimes}_k} A_0$ and $\varphi: A \to A'$ is a local homomorphism reducing to id_{A_0} (mod m_{A_1}) then $E(\varphi): E(A) \to E(A')$ is the map $E(\varphi)([A[[t_A{}^n, y(t_A)]]]) = [A' \mathbin{\widehat{\otimes}_A} (A[[t_A{}^n, y(t_A)]])]$. Let $E(B_{A_0}) = E(A_0[\epsilon])$. As usual in this theory, $E(A_0)$ is an A_0-module (see the proof of the following Theorem). It will be assumed that the notion of a *versal deformation* is familiar.

Recall that there are commutative algebra cohomology groups $T^i(B/A, M) = H^i(\mathrm{Hom}(L_{B/A}, M))$ where $L_{B/A}$ is the cotangent complex of the A-algebra B and M is a B-module. If B is a relative complete intersection $T^i(B/A, M) = 0$ for $i > 1$: in particular this is true if B is a plane branch over A and A is a power series ring. If B_A is a plane branch and $R_A = A[[X, Y]]$ and $0 \to (f) \to R_A \to B_A \to 0$ then $T^i(B_A/A, M)$, $i = 0, 1$ are the cohomology groups of the complex

$$(f)/(f^2) \to \Omega_{R_A/A} \otimes_{R_A} B_A \to \Omega_{B_A/A} \to 0$$

so that $T^0(B_A/A, M) = \mathrm{Der}_A(B_A, M)$ and there is a presentation

$$\mathrm{Hom}(\Omega_{R_A/A} \otimes_{R_A} B_A, M) \to \mathrm{Hom}((f)/(f^2), M) \to T^1(B_A/A, M) \to 0.$$

The latter map is realized as follows. If $g: (f)/(f^2) \to M$ is a B_A-module map, the push-out in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (f)/(f^2) & \longrightarrow & R_A/(f^2) & \longrightarrow & B_A & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle g} & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & M & \longrightarrow & g_*(R_A/(f^2)) & \longrightarrow & B_A & \longrightarrow & 0
\end{array}$$

is an element of $Ex^1(B_A/A, M)$, the set of isomorphism classes of square-zero algebra extenstions of B_A by M; in this way one gets a B_A-module isomorphism $T^1(B_A/A, M) \to Ex^1(B_A/A, M)$. If $i: M \to N$ is a B_A-module homomorphism the map $i^*: T^1(B_A/A, M) \to T^1(B_A/A, N)$ induced on the cohomology is realized by an exactly similar push-out of algebra extensions. If $E \in T^1(B_A/A, M)$ and m_{B_A} is the maximal ideal of B_A (with the prevailing assumptions about completeness B_A is local) E is represented by an algebra extension E: $0 \to M \to$

$X \to B_A \to 0$ which by the above isomorphism arises by push-out via a B_A-module map $g\colon (f)/(f^2) \to M$. E is *section-preserving* if g factors $g\colon (f)/(f^2) \to m_{B_A}M \subset M$. The set of section-preserving $E \in T^1(B_A/A, M)$ forms a submodule denoted $T^1_{\mathrm{sec}}(B_A/A, M)$ which is clearly the image of $\mathrm{Hom}((f)/(f^2), m_{B_A}M)$. Thus for a plane branch B_A, $T^1_{\mathrm{sec}}(B_A/A, M) = m_{B_A}M/(m_{B_A}M) \cap (J_AM)$ where $J_A = (f_X, f_Y)$ is the jacobian ideal of B_A. As long as B_A is singular ($n \geqslant 2$), $J_A \subset m_{B_A}$, so $T^1_{\mathrm{sec}}(B_A/A, M) = m_{B_A}M/J_AM$.

In general if A is a domain, K the quotient field of A, and C is a complete K-algebra, C is *integral* if there is an A-algebra C' and a K-algebra isomorphism $C \to K \,\hat{\otimes}_A\, C'$. *Every* parametrized branch over a power series ring is integral in this sense. In the following Theorem, the base rings are *assumed* to be integral, that is, projective limits of artin K-algebras of the form $A_1 = (\mathbf{A}_1 \otimes_k A_0) \otimes_{A_0} K = \mathbf{A}_1 \otimes_k K$ where $\mathbf{A}_1$ is an artin k-algebra.

THEOREM 2.3. *Let $B_{A_0} = A_0[[t^n, y(t)]] = A_0[[X, Y]]/(f)$ be a parametrized branch over a power series ring $A_0 = k[[S_1, \ldots, S_t]]$, K the quotient field of A_0, and $B_K = K \,\hat{\otimes}_{A_0} B_{A_0}$. There is an integral versal E-deformation $B_{K[[T_1,\ldots,T_r]]} = K \,\hat{\otimes}_{A_0} A_0[[T_i, t^n, y_T(t)]]$ of B_K, where $r = \dim_K E(K[\epsilon]) < \infty$, and*

$$E(K[\epsilon]) = \mathrm{Ker}(T^1_{\mathrm{sec}}(B_K/K, B_K) \to T^1_{\mathrm{sec}}(B_K/K, B(I, t)_K)$$
$$= ((J_K m_{B(I,t)_K}) \cap m_{B_K})/J_K$$

where the map on cohomology groups is induced by the inclusion $i\colon B_K \to B(I, t)_K$ and $J_K = (f_X, f_Y)B_K$ is the jacobian ideal of B_K.

Remark 2.4. It is not possible to prove the Theorem for E-deformations of B_{A_0} because in general $E(A_0[\epsilon])$, as an A_0-module, has infinitely generated torsion submodule.

Proof of Theorem 2.3. An element of $T^1_{\mathrm{sec}}(B_K/K, B_K)$ can be regarded as an algebra over $K[\epsilon]$; let $[B_{K[\epsilon]}] \in \mathrm{Ker}(T^1_{\mathrm{sec}}(B_K/K, B_K) \xrightarrow{i_*} T^1_{\mathrm{sec}}(B_K/K, B(I, t)_K))$, (where the square brackets denote "isomorphism class"). Then there is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_K & \longrightarrow & B_{K[\epsilon]} & \longrightarrow & B_K & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & \| & & \\ 0 & \longrightarrow & B(I, t)_K & \longrightarrow & i_*(B_{K[\epsilon]}) & \longrightarrow & B_K & \longrightarrow & 0 \end{array}$$

where by assumption the lower extension splits. From the definition of the push-out it is easy to see that j is a monomorphism so $j(B_{K[\epsilon]})$, hence $B_{K[\epsilon]}$, can be regarded as a subalgebra of $K[\epsilon] \otimes_K B(I, t)_K$. Then

$$B_{K[\epsilon]} = K[\epsilon][[t^n + \epsilon x_1, y(t) + \epsilon y_1]] \subset K[\epsilon] \otimes_K B(I, t)_K .$$

Where, because the deformation is section-preserving, $x_1, y_1 \in m_{B(I,t)_K}$. Then by Hensel's lemma $t^n + \epsilon \cdot x_1 = t^n_{K[\epsilon]}$ for some parameter $t_{K[\epsilon]}$ in $K[\epsilon][[t]]$ and $y(t) + \epsilon \cdot y_1 = y_2(t_{K[\epsilon]})$. One checks that

$$B_{K[\epsilon]} = K[[\epsilon, t^n_{K[\epsilon]}, y_2(t_{K[\epsilon]})]] \subset K[\epsilon] \otimes_K B(I, t_{K[\epsilon]})_K :$$

hence $B_{K[\epsilon]}$ is an E-deformation.

If $B_{K[\epsilon]} = K[[\epsilon, t^n_{K[\epsilon]}, y_2(t_{K[\epsilon]})]] \subset K[\epsilon] \otimes_K B(I, t)_K$ is an E-deformation of B_K then since $B_{K[\epsilon]}$ is $K[\epsilon]$-flat by Lemma 1.5(iii) there is an algebra extension

$$0 \to B_K \to B_{K[\epsilon]} \to B_K \to 0$$

i.e. $[B_{K[\epsilon]}] \in T^1_{\mathrm{sec}}(B_K/K, B_K)$.

Let $p\colon R_K = K[[X, Y]] \to B_K$ be given by $p(X) = t^n$, $p(Y) = y(t)$. By smoothness p lifts to $q\colon R_K \to B_{K[\epsilon]}$ with $q(x) = t^n_{K[\epsilon]}$, $q(Y) = y_2(t_{K[\epsilon]})$. Using the K-vector space splitting of the algebra extension one can write $q = p + \epsilon \cdot D$, where $D \in \mathrm{Der}_K(R_K, B(I, t)_K) = \mathrm{Hom}_{R_K}(\Omega_{R_K/K}, B(I, t)_K)$ and D is induced from a $D' \in \mathrm{Hom}_{B_K}(\Omega_{R_K/K} \otimes_{R_K} B_K, B(I, t)_K)$ since D is a derivation into a B_K-module. But then recalling the presentation of $T^1(B/A, M)$ above, the image of D' in $T^1_{\mathrm{sec}}(B_K/K, B(I, t)_K)$ is $i_*([B_{K[\epsilon]}])$, so $i_*([B_{K[\epsilon]}]) = 0$, and $[B_{K[\epsilon]}] \in \mathrm{Ker}(T^1_{\mathrm{sec}}(B_K/K, B_K) \to T^1_{\mathrm{sec}}(B_K/K, B(I, t)_K))$. The formula of the Theorem follows since $T^1_{\mathrm{sec}}(B_K/K, M) = (m_{B_K} M)/J_K M$ for any B_K-module M (again by the presentation of T^1), by a Noether isomorphism.

The existence of the versal E-deformation (equivalently: of a "hull" for the functor E: (Artin K-algebras) $\to$ (Sets)) is a standard verification of the conditions H_1, H_2, H_3 of Schlessinger [7, p. 212].

It is convenient to check H_3 first: $\dim_K E(K[\epsilon]) = \dim_K T^1(B_K/K, B_K) < \infty$ because $\mathrm{Spec}(B_K)$ has an isolated singularity. Let $E_1, \ldots, E_r \in E(K[\epsilon])$ be a K-basis, where $E_i = [B_{K[\epsilon],i}]$ and $B_{K[\epsilon],i} = K[[\epsilon, t^n_{K[\epsilon],i}, y_{1,i}(t_{K[\epsilon],i})]] = K[[\epsilon, t^n + \epsilon \cdot x_i, y(t) + \epsilon \cdot y_i]]$ where $x_i, y_i \in m_{B(I,t)_K}$. Now B_K is finitely determined because $\mathscr{Q}_{B_K/B_K} = (t^N)\, K[[t]] \subset B_K$, and this implies that $B_{K[\epsilon],i}$ is finitely determined for $i = 1, \ldots, r$, i.e. that one can assume that $x_i, y_i \in K[t]$. Then there is an $a \in A_0 = k[[S_1, \ldots, S_t]]$ with $ax_i, ay_i \in A_0[t]$ and then $aE_i = [K[[\epsilon, t^n + ax_i, y(t) + ay_i]]]$ is integral. Further, writing $t^n + \epsilon \cdot ax_i = t^n_{K[\epsilon],a,i}$, $y(t) + \epsilon a y_i = y_{2,i}(t_{K[\epsilon],a,i})$ is an integral operation, i.e. one can choose $t_{K[\epsilon],a,i} \in A_0[[\epsilon, t]]$. Thus $E(K[\epsilon])$ has a K-basis consisting of integral E- deformations.

Now let

$$\begin{tikzcd} & A_1' \times_{A_1} A_1'' \arrow[dl, "\pi'"'] \arrow[dr, "\pi''"] & \\ A_1' \arrow[dr, "p'"'] & & A_1'' \arrow[dl, "p''"] \\ & A_1 & \end{tikzcd}$$

be a fiber product of artin K-algebras with $A_1 = K \otimes_k \mathbf{A}_1$ where $\mathbf{A}_1$ is an artin k-algebra, etc., and consider the natural map

$$\varphi\colon E(A_1' \times_{A_1} A_1'') \to E(A_1') \times_{u(A_1)} E(A_1'').$$

One has to check that φ is surjective if $A_1'' \to A_1$ is a *small extension* (a surjection with one-dimensional kernel), H_1; and that φ is bijective if $A_1 = K$ and $A_1'' = K[\epsilon]$, H_2.

Let $B_{A_1'} = A_1'[[t^n + x_1', y(t) + y_1']]$, $B_{A_1''} = A_1''[[t^n + x_1'', y(t) + y_1'']]$, $B_{A_1} = A_1[[t^n + x_1, y(t) + y_1]]$ with $A_1 \otimes_{A_1'} B_{A_1'} = B_{A_1} = A_1 \otimes_{A_1''} B_{A_1''}$ be integral E-deformations of B_K i.e. $x_1', y_1' \in (A_0 \otimes_k m_{\mathbf{A}_1'}) \widehat{\otimes}_k m_{B(I,t)}$ etc. Then composing the above isomorphisms with automorphisms of B_{A_1} one can assume that $p'(x_1') = x_1 = p''(x_1'')$, $p'(y_1') = y_1 = p''(y_1'')$. Choose $\mathbf{x}_1, \mathbf{y}_1 \in (A_0 \otimes_k m_{\mathbf{A}_1'} \times_{\mathbf{A}_1} \mathbf{A}_1'') \widehat{\otimes}_k m_{B(I,t)}$ with $\pi'(\mathbf{x}_1) = x_1'$, $\pi''(\mathbf{x}_1) = x_1''$, $\pi'(\mathbf{y}_1) = y_1'$, $\pi''(\mathbf{y}_1) = y_{''}^1$. Then if $B_{A_1'} \times_{A_1} A_1'' = (A_1' \times_{A_1} A_1'')[[t^n + \mathbf{x}_1, y(t) + \mathbf{y}_1]]$, $\varphi([B_{A_1'} \times_{A_1} A_1'']) = [B_{A_1'}] \times_{[B_{A_1}]} [B_{A_1''}]$ and φ is surjective. As before one checks that $B_{A_1'} \times_{A_1} A_1''$ can be regarded "integrally" as an E-deformation by writing $t^n + \mathbf{x}_1 = t^n_{A_1'} \times_{A_1} A_1''$ where $t_{A_1'} \times_{A_1} A_1'' \in (A_0 \otimes_k m_{\mathbf{A}_1'} \times_{\mathbf{A}_1} \mathbf{A}_1'')[[t]]$, and then $B_{A_1'} \times_{A_1} A_1'' \subset (A_1' \times_{A_1} A_1'') \otimes_k B(I, t_{A_1'} \times_{A_1} A_1'')$.

The same argument with $A_1' = K$ and $A_1'' \to A_1$ a small extension shows that there is no obstruction to lifting E-deformations (as flat deformations, by the pervasive Lemma 1.5(iii), and therefore that the functor $E(\_)$ is *smooth*, i.e. that the versal hull must be a power series ring $K[[T_i]]$.

To check H_2, notice that $A_1' \times_K K[\epsilon] \to A_1' \to 0$ is a split algebra extension and that this splitting makes the lifting of an element of $E(A_1') \times_{pt} E(K[\epsilon])$ to $E(A_1' \times_K K[\epsilon])$ defined above into an inverse of φ: so φ is bijective in this case. Now applying Schlessinger's Theorem one obtains an integral versal E-deformation $B_{K[[T_1,\ldots,T_k]]} = K[[T_i, t^n_{K[[T_i]]}, y_{2,T}(t_{K[[T_i]]})]] = K[[T_i, t^n, y_T(t)]]$ (the last isomorphism coming from a final parameter change), as asserted.

References

1. K. Brauner, Zur Geometrie der Functionen zweier komplexen Veränderlichen, III, IV, *Abh. Math. Sem. Univ. Hamburg* **6** (1928), 8–54.
2. C. Delorme, Sur les modules des singularités des courbes planes, Orsay notes, 1975.
3. M. Merle, Sur l'espace des modules des branches singulières, Pub. du Centre de Mathématiques de l'École Polytechnique, 1977.
4. J. Milnor, "Singular Points of Complex Hypersurfaces," Annals of Math. Studies No. 61, Princeton Univ. Press, Princeton, N. J., 1968.
5. A. Nobile, On equisingular families of plane branches, Louisiana State University notes, 1972.
6. F. Pham, Singularités des courbes planes: Une introduction à la géométrie analytique complex, Pub. du Centre de Mathématiques de l'École Polytechnique, Paris 1970.
7. M. Schlessinger, Functors of Artin rings, *Trans. Amer. Math. Soc.* **130** (1968), 208–222.

8. B. TESSIER, Introduction to equisingularity problems, *in* "Algebraic Geometry-Arcata 1974," Amer. Math. Soc. Proceedings of Symposia in Pure Math., Vol. 29, Providence, R. I., 1975.
9. J. WAHL, Equisingular deformations of plane algebroid curves, *Trans. Amer. Math. Soc.* **193** (1974), 143–170.
10. O. ZARISKI, On the topology of algebroid singularities, *Amer. J. Math.* **54** (1932), 453–465.
11. O. ZARISKI, Studies in equisingularity, I, *Amer. J. Math.* **87** (1965), 507–536; II, *Amer. J. Math.* **87** (1965), 972–1006; III. Saturation of local rings and equisingularity, *Amer. J. Math.* **90** (1968), 961–1023.
12. O. ZARISKI, Le problème des modules pour les branches planes, Pub. du Centre de Mathématiques de l'École Polytechnique, 1976 (notes by F. Kmety and M. Merle; with an appendix (A) by B. Tessier).